3. Butenin, N. V. and Lestev, A. I., On the theory of the gyrosemicompass. Izv. Vyssh. Uch. Zav. Priborostroenie Vol. 7, No. 4, 1964.
4. Merkin, D. R., Gyroscopic Systems. Gostekhizdat, Moscow, 1956.
5. Bunatian, L. A., The motion of a free gyroscope on a movable base with allowance for the mass of the rings of the universal suspension. Dokl. Akad. Nauk SSSR Vol. 152, No. 2, 1963.

Translated by A. Y.

APPLICATION OF THE LIAPUNOV METHOD TO LINEAR SYSTEMS WITH LAG

PMM Vol. 31, No. 5, 1967, pp. 959-963
V. B. KOLMANOVSKII
(Moscow)
(Received April 3, 1966)

Several authors have investigated the properties of the solutions of differential equations of the form

$$x^{\cdot}(t) = -\int_0^\infty x(t-s)\, dK_0(t, s) \tag{1}$$

$$x^{\cdot\cdot}(t) = -\int_0^\infty x^{\cdot}(t-s)\, dK_1(t, s) - \int_0^\infty x(t-s)\, dK_2(t, s) \tag{2}$$

Here and below the symbol $dK_i(t, s)$ denotes the differential with respect to the second argument. Attention has been largely confined to the case of concentrated lag, i.e. to step functions $K_i(t, s)$. The general case of lag distributed over a finite interval $[0, s(t)]$ was first investigated by Myshkis [1].

Stability conditions for the solutions of differential equations of this type were obtained in [2] under the assumption that the kernels K_i depend only on s, i.e. that $K_i(t, s) \equiv K_i(s)$ and that the variation of the functions $K_i(s)$ in $[0, \infty)$ is bounded.

The present paper concerns the stability conditions for trivial solutions of equations of the form (1) and (2).

We assume that the functions $K_i(t, s)$ satisfy the following requirement of bounded variation with respect to s:

$$\sup_t \int_0^\infty |dK_i(t, s)| \leqslant \text{const} < \infty \qquad (0 \leqslant t < \infty) \tag{3}$$

The solution $x(t)$ of Eq. (1) (Eq. (2)) for $t > 0$ is determined by the function $\varphi(t)$ (the function $\psi(t)$) specified on $(-\infty, 0]$,

$$x(t) = \varphi(t), \qquad t \leqslant 0 \tag{4}$$

$$(x(t) = \psi(t), \ x^{\cdot}(t) = \psi^{\cdot}(t), \ t \leqslant 0) \tag{5}$$

Definition. The solution $x_1(t)$ of problem (1), (4) (the solution $x_2(t)$ of problem (2), (5)) will be called "stable" if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$|x_1(t)| < \varepsilon \quad \text{for} \ t \geqslant 0 \qquad (|x_2(t)| < \varepsilon \quad \text{for} \ t \geqslant 0)$$

as soon as

$$\|\varphi\|_1 = \sup_{\theta\leqslant 0}|\varphi(\theta)| < \delta(\varepsilon) \quad (\|\psi\|_2 = \sup_{\theta\leqslant 0}(|\varphi(\theta)| + |\psi^{\cdot}(\theta)|) < \delta(\varepsilon))$$

If, furthermore, $x_i(t) \to 0$ as $t \to \infty$, $i = 1, 2$, then the solution will be termed "asymptotically stable".

The general theorems of Liapunov's second method as applied to equations with infinite lag can be verified by repeating verbatim the proofs of the corresponding theorems for the case of finite lag. A similar remark was already made in [3]. For this reason we shall set down without proof a theorem which is a generalization of Theorem 31.3 of [4] and is necessary in our investigation. Let us consider Eq.

$$x^{\cdot}(t) = F(x(t+\theta), t) \tag{6}$$

defined on the n-dimensional space E_n, where $F(x(\theta), t)$ for fixed t is a continuous functional defined on the functions $x(\theta)$ of the argument θ varying in the range $-\infty < \theta \leq 0$; for fixed $x(\theta)$ the function $F(x(\theta), t)$ is continuous with respect to t.

Let us assume that $F(\varphi, t) \equiv 0$ for $\varphi \equiv 0$, and that

$$|F(x(\theta), t) - F(y(\theta), t)| \leqslant L\|x - y\| = L \sup_{\theta\leqslant 0} \sum_{i=1}^{n} |x_i(\theta) - y_i(\theta)|$$

The symbol dV/dt will be used to represent

$$\overline{\lim_{\Delta t \to +0}} \frac{1}{\Delta t}[V(x(t+\theta+\Delta t), t+\Delta t) - V(x(t+\theta), t)]$$

where $x(t)$ is a solution of system (6).

Theorem 1 The trivial solution of Eq. (6) is asymptotically stable if there exists a continuous functional $V(x(\theta), t)$ which satisfies the estimates

$$V(x(\theta), t) \leqslant W_1(|x(0)|) + W_2(\|x(\theta)\|), \qquad W_1(0) = W_2(0) = 0 \tag{7}$$

$$V(x(\theta), t) \geqslant \omega(|x(0)|), \qquad |x(0)| = (\Sigma_i^2(0))^{1/2} \tag{8}$$

$$dV / dt \leqslant -f(|x(0)|) \tag{9}$$

Here the continuous functions $W_1(r)$, $W_2(r)$ are monotonous for $r \geq 0$, while the continuous functions $\omega(r)$ and $f(r)$ are positive for $r > 0$.

Note. The trivial solution of Eq. (6) is stable if conditions (7) and (8) of the above Theorem are fulfilled and of the following condition is fulfilled instead of condition (9):

$$dV / dt \leqslant 0 \tag{10}$$

Theorem 2. Let the kernel $K_0(t, s)$ experience a jump of magnitude $K_0(t, 0)$ at zero, and let this jump satisfy the condition

$$\inf_t \left[2K_0(t, 0) - \int_{+0}^{\infty} |dK_0(t, s)| - \int_{+0}^{\infty} |dK_0(t+s, s)|\right] > 0 \qquad (t \geqslant 0) \tag{11}$$

The trivial solution of problem (1), (6) is then asymptotically stable if

$$\sup_t \int_{+0}^{\infty} \int_{t-s}^{t} |dK_0(\tau+s, s)| \, d\tau \leqslant C < \infty$$

Proof. Let us consider the functional

$$V(x(t+\theta), t) = x^2(t) + \int_{+0}^{\infty} \int_{t-s}^{t} |dK_0(\tau+s, s)| \, x^2(\tau) \, d\tau \tag{12}$$

The derivative of this functional with respect to t along the trajectory of system (1) is

$$\frac{dV(x(t+\theta),t)}{dt}\leqslant -x^2(t)\left[2K_0(t,0)-\int_{+0}^{\infty}|dK_0(t,s)|-\int_{+0}^{\infty}|dK_0(t+s,s)|\right]$$

By virtue of Theorem 1, this fact together with (11) and (12) proves Theorem 2.

Corollary. Theorem 2 remains valid if we replace condition (11) by

$$\inf_t K_0(t,0)>\int_{+0}^{\infty}\sup_t|dK_0(t,s)|\qquad (t\geqslant 0)$$

Let us set

$$K_{ij}(t,s)=\frac{1}{2}\left(\int_0^s|dK_i(t,\tau)|-(-1)^jK_i(t,s)\right)\qquad (j=1,2)$$

Then

$$K_i(t,s)=K_{i1}(t,s)-K_{i2}(t,s)\qquad (i=0,1,2)\tag{13}$$

Here $K_{ij}(t,s)$ increase monotonously with respect to s for each fixed value of $t>0$. Here we assume that $K_{ij}(t,s)\equiv 0$ for $t<0$.

On the basis of (13) Eqs. (1) and (2) can be rewritten as

$$x^{\cdot}(t)=-\int_0^{\infty}x(t-s)\,dK_{01}(t,s)+\int_0^{\infty}x(t-s)\,dK_{02}(t,s)\qquad x^{\cdot}(t)=y(t)\tag{14}$$

$$y^{\cdot}(t)=-\int_0^{\infty}y(t-s)\,dK_{11}(t,s)+\int_0^{\infty}y(t-s)\,dK_{12}(t,s)-$$
$$-\int_0^{\infty}x(t-s)\,dK_{21}(t,s)+\int_0^{\infty}x(t-s)\,dK_{22}(t,s)\tag{15}$$

Theorem 3. Let the conditions

$$\sup_t\int_0^{\infty}\int_{t-s}^{t}dK_{01}(\tau+s,s)\,d\tau<1\qquad (t\geqslant 0)\tag{16}$$

$$\inf_t\left[\int_0^{\infty}dK_{01}(t+s,s)-\int_0^{\infty}dK_{02}(t,s)-2\int_0^{\infty}dK_{02}(t+s,s)-\right.$$
$$\left.-\int_0^{\infty}s\,dK_{01}(t+s,s)\sup_t\int_0^{\infty}dK_{02}(t,s)-\int_0^{\infty}s\,dK_{01}(t+s,s)\sup_t\int_0^{\infty}dK_{01}(t,s)\right]>0\qquad (t\geqslant 0)$$

be fulfilled.

Then the trivial solution of problem (1), (4) is asymptotically stable provided that

$$\int_0^{\infty}\int_0^{\infty}\int_{t-s_1}^{t}dK_{01}(\tau+s_1,s_1)\,d\tau\int_{\tau}^{t}dK_{0j}(\tau_1,s)\,d\tau_1\leqslant c<\infty$$

$$\int_0^{\infty}\int_{t-s}^{t}d\tau\int_{\tau}^{t}dK_{01}(\tau_1+s,s)\,d\tau_1\leqslant c\qquad \int_0^{\infty}\int_{t-s}^{t}dK_{02}(\tau+s,s)\,d\tau\leqslant c\tag{17}$$

Proof. Let us consider the functional

$$V(x(t+\theta),t)=x^2(t)\left[1-\int_0^{\infty}\int_{t-s}^{t}dK_{01}(\tau+s,s)\,d\tau\right]+$$
$$+\int_0^{\infty}\int_{t-s}^{t}[x(t)-x(\tau)]^2\,dK_{01}(\tau+s,s)\,d\tau+$$

$$+\int_0^\infty\int_0^\infty\int_{t-s_1}^t dK_{01}(\tau+s_1,\, s_1)\,d\tau\int_\tau^t [x(\tau)-x(\tau_1-s)]^2\,d\tau_1\,dK_{01}(\tau_1,\, s)\,+$$
$$+\int_0^\infty\int_0^\infty\int_{t-s_1}^t dK_{01}(\tau+s_1,\, s_1)\,d\tau\int_\tau^t [x(\tau)+x(\tau_1-s)]^2\,d\tau_1\,dK_{02}(\tau_1,\, s)\,+$$
$$+\int_0^\infty\int_{t-s}^t [x(\tau)-x(\tau-s)]^2\,dK_{02}(\tau,\, s)+2\int_0^\infty\int_{t-s}^t x^2(\tau)\,dK_{02}(\tau+s,\, s)\,d\tau\,+$$
$$+\int_0^\infty\int_{t-s}^t d\tau\int_\tau^t x^2(\tau_1)\,dK_{01}(\tau_1+s,\, s)\,d\tau_1\,\sup_t\int_0^\infty dK_{02}(t,\, s)\,+$$
$$+\int_0^\infty\int_{t-s}^t d\tau\int_\tau^t x^2(\tau_1)\,dK_{01}(\tau_1+s,\, s)\,d\tau_1\,\sup_t\int_0^\infty dK_{01}(t,\, s)\qquad (t\geqslant 0)\tag{18}$$

The derivative of functional (18) with respect to t satisfies the inequality

$$\frac{dV(x(t+\theta),\, t)}{dt}\leqslant-\int_0^\infty x^2(t-s)\,|\,dK_0(t,\, s)\,|\left[1-\int_0^\infty\int_{t-s_1}^t dK_{01}(\tau+s_1,\, s_1)\,d\tau\right]-$$
$$-\,x^2(t)\left[\int_0^\infty dK_{01}(t+s,\, s)-\int_0^\infty dK_{02}(t,\, s)-2\int_0^\infty dK_{02}(t+s,\, s)\,-\right.$$
$$\left.-\int_0^\infty s\,dK_{01}(t+s,\, s)\left(\sup_t\int_0^\infty dK_{01}(t,\, s)+\sup_t\int_0^\infty dK_{02}(t,\, s)\right)\right]\qquad (t\geqslant 0)$$

along the trajectories of system (15).

By virtue of Theorem 1, conditions (16), (17) of Theorem 3 and (18) imply the validity of Theorem 3.

Theorem 4. Let the kernel $K_1(t, s)$ of Eq. (15) have a jump of magnitude $K_{11}(t,\, 0)$ at zero and let the conditions

$$\inf_t\int_0^\infty dK_2(t,\, s)>0\qquad (t\geqslant 0)\tag{19}$$
$$\sup_t\,\overline{\lim_{\Delta t\to+0}}\left[\int_0^\infty dK_2(t+\Delta t,\, s)-\int_0^\infty dK_2(t,\, s)\right]\leqslant 0\tag{20}$$
$$\inf_t\left[2K_{11}(t,\, 0)-2\int_{+0}^\infty |\,dK_1(t,\, s)\,|-\int_0^\infty s\,|\,dK_2(t,\, s)\,|-\int_0^\infty\int_{t-s}^t d\tau\,|\,dK_2(\tau+s,\, s)\,|\right]\geqslant 0\tag{21}$$

be fulfilled.

Then the trivial solution of problem (2), (5) is asymptotically stable if

$$\int_{+0}^\infty\int_{t-s}^t |\,dK_1(\tau+s,\, s)\,|\,d\tau\leqslant C<\infty$$
$$\int_0^\infty\int_{t-s}^t d\tau\int_\tau^t |\,dK_{2j}(\tau_1,\, s)\,|\,d\tau_1\leqslant C\qquad (j=1,2)$$
$$\int_0^\infty\int_{t-s}^t (t-\tau)\,|\,dK_2(\tau+s,\, s)\,|\,d\tau\leqslant C$$

Proof. Let us introduce the functional

$$V(x(t+\theta), y(t+\theta), t) = \int_{+0}^{\infty} \int_{t-s}^{t} |dK_1(\tau+s, s)| y^2(\tau) d\tau + \\ + \int_0^{\infty} \int_{t-s}^{t} d\tau \int_{\tau}^{t} [y(\tau) - y(\tau_1)]^2 d\tau_1 dK_{21}(\tau_1, s) + \\ + \int_0^{\infty} \int_{t-s}^{t} d\tau \int_{\tau}^{t} [y(\tau) + y(\tau_1)]^2 d\tau_1 dK_{22}(\tau_1, s) + \int_0^{\infty} \int_{t-s}^{t} |dK_2(\tau+s, s)| d\tau \int_{\tau}^{t} y^2(\tau_1) d\tau_1 + \\ + y^2(t) + x^2(t) \int_0^{\infty} dK_2(t, s) \qquad (22)$$

We have

$$dV/dt \leqslant -y^2(t)\Big[2K_{11}(t, 0) - 2\int_{+0}^{\infty} |dK_1(t, s)| - \\ - \int_0^{\infty} s\,|dK_2(t, s)| - \int_0^{\infty} \int_{t-s}^{t} d\tau\, |dK_2(\tau+s, s)|\Big] + \\ + x^2(t) \varlimsup_{\Delta t \to +0} \Big[\int_0^{\infty} dK_2(t+\Delta t, s) - \int_0^{\infty} dK_2(t, s)\Big]$$

By virtue of Theorem 1, this together with the conditions of Theorem 4 and (22) implies the validity of Theorem 4.

The devices used in constructing functionals (18) and (22) make it possible to obtain the stability conditions for the solutions of Eq. (2) in the general case (i.e. in the case of kernels without jumps) with the aid of simple but cumbersome computations.

Example 1. The system

$$x^{\cdot}(t) = -a(t)\,x(t) + b(t)\,x(t-\tau) \qquad (\tau \geqslant 0)$$

is asymptotically stable by virtue of Theorem 2, provided that

$$\inf_t a(t) > \sup_t |b(t)| \qquad (t \geqslant 0)$$

Example 2. The system

$$x^{\cdot}(t) = a(t)\,x(t) - b(t)\,x(t-\tau) \qquad (a(t) > 0,\ b(t) > 0,\ \tau \geqslant 0)$$

is asymptotically stable by virtue of Theorem 3 if

$$\sup_t \int_{t-\tau}^{t} b(\tau+s)\,ds < 1 \qquad (t \geqslant 0)$$

$$\inf_t b(t) > \sup_t (3a(t) + \tau b(t+\tau) \sup_t a(t) + \tau b(t+\tau) \sup_t b(t))$$

The author is grateful to R. Z. Khas'minskii for formulating the problem and for his interest in the present study.

BIBLIOGRAPHY

1. Myshkis, A. D., Linear Differential Equations with a Lagging Argument. Gostekhizdat, Moscow-Leningrad, 1951.
2. Kolmanovskii, V. B. and Khas'minskii, R. Z., The stability of linear systems with lag. Izv. Vyssh. Uch. Zav., Mat. No. 4, 1966.
3. Hale, G., Sufficient conditions for stability and instability of autonomous functional-differential equations. J. Diff. Equat., Vol. 1, No. 4, 1965.
4. Krasovskii, N. N., Some problems in Motion Stability Theory, Fizmatgiz, 1959.

Translated by A. Y.